Integration by parts and more difficult techniques

1. (a) $$\int \sec^4 x dx = \int (1+\tan^2 x) d(\tan x) = \tan x + \frac{\tan^3 x}{3} + C$$

(b) $$\int \sec^n x dx = \int \sec^{n-2} x d(\tan x) = \sec^{n-2} x \tan x - \int \tan x d(\sec^{n-2} x)$$

$$= \sec^{n-2} x \tan x - \int \tan x \cdot (n-2) \sec^{n-3} x (\sec x \tan x) dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \tan^2 x \cdot \sec^{n-2} x dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int (\sec^2 x - 1) \cdot \sec^{n-2} x dx$$

$$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$\therefore I_n = \frac{1}{n-1}\left[\sec^{n-2} x \tan x + (n-2) I_{n-2}\right]$$

$$\therefore I_5 = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4} I_3 = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\left[\frac{1}{2}(\sec x \tan x + I_1)\right]$$

$$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] + C$$

(c) $$\int \ln x dx = x \ln x - \int x d(\ln x) = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int dx = x \ln x - x + C$$

(d) $$\int x a^x dx = \int x e^{x \ln a} dx = \frac{1}{\ln a} \int x d(e^{x \ln a}) = \frac{1}{\ln a}\left[x e^{x \ln a} - \int e^{x \ln a} dx\right] = \frac{1}{\ln a}\left[x e^{x \ln a} - \frac{1}{\ln a} e^{x \ln a}\right] + C$$

$$= \frac{1}{(\ln a)^2}\left[x a^x \ln a - a^x\right] + C = \frac{a^x}{(\ln a)^2}\left[x \ln a - 1\right] + C$$

(e) $$\int (e^x + 2x)^2 dx = \int (e^{2x} + 4xe^x + 4x^2) dx = \int e^{2x} dx + 4\int x d(e^x) + 4\int x^2 dx$$

$$= \frac{1}{2}e^{2x} + 4\left[xe^x - \int e^x dx\right] + \frac{4}{3}x^3 + C = \frac{1}{2}e^{2x} + 4xe^x - 4e^x + \frac{4}{3}x^3 + C$$

(f) $$\int \frac{\ln x}{(1+x)^2} dx = \int \ln x d\left(-\frac{1}{1+x}\right) = -\frac{\ln|x|}{1+x} - \int \left(-\frac{1}{1+x}\right) d(\ln x) = = -\frac{\ln|x|}{1+x} + \int \frac{1}{1+x}\frac{dx}{x}$$

$$= -\frac{\ln|x|}{1+x} + \int \left(\frac{1}{x} - \frac{1}{1+x}\right) dx = -\frac{\ln|x|}{1+x} + \ln|x| - \ln|1+x| + C$$

(g) $$\int \frac{\ln(x+1)}{\sqrt{x+1}} dx = 2\int \ln(x+1)\ d\left(\sqrt{x+1}\right) = 2\left[\sqrt{x+1}\ln\sqrt{x+1} - \int \sqrt{x+1}\ d(\ln(x+1))\right]$$

$$= 2\left[\sqrt{x+1}\ln\sqrt{x+1} - \int \sqrt{x+1}\ \frac{dx}{x+1}\right] = 2\left[\sqrt{x+1}\ln\sqrt{x+1} - \int \frac{dx}{\sqrt{x+1}}\right] = 2\left[\sqrt{x+1}\ln\sqrt{x+1} - 2\sqrt{x+1}\right] + C$$

(h) $$I = \int e^x \cos x dx = \int \cos x d(e^x) = e^x \cos x - \int e^x d(\cos x) = e^x \cos x + \int e^x \sin x dx = e^x \cos x + \int \sin x d(e^x)$$

$$= e^x(\sin x + \cos x) - \int e^x \cos x dx = e^x(\sin x + \cos x) - I$$

$$\therefore I = \frac{1}{2}e^x(\sin x + \cos x) + C$$

(i) $$I = \int e^{-x/2}\cos 2x dx = -2\int \cos 2x d(e^{-x/2}) = -2\left[e^{-x/2}\cos 2x - \int e^{-x/2} d(\cos 2x)\right] = -2\left[e^{-x/2}\cos 2x + 2\int e^{-x/2}\sin 2x dx\right]$$

$$= -2\left[e^{-x/2}\cos 2x - 4\int \sin 2x d(e^{-x/2})\right] = -2e^{-x/2}\cos 2x + 8\left[e^{-x/2}\sin 2x - \int e^{-x/2} d(\sin 2x)\right]$$

$$= -2e^{-x/2}\cos 2x + 8e^{-x/2}\sin 2x - 16I \qquad \therefore I = -\frac{2}{17}\left[e^{-x/2}\cos 2x - 4e^{-x/2}\sin 2x\right] + C$$

1. (j) $\int \sqrt{x}\ln x dx = \frac{2}{3}\int \ln x d\left(x^{\frac{3}{2}}\right) = \frac{2}{3}\left[x^{\frac{3}{2}}\ln x - \int x^{\frac{3}{2}} d(\ln x)\right] = \frac{2}{3}\left[x^{\frac{3}{2}}\ln x - \int x^{\frac{3}{2}}\frac{dx}{x}\right]$

$= \frac{2}{3}\left[x^{\frac{3}{2}}\ln x - \int x^{\frac{1}{2}} dx\right] == \frac{2}{3}\left[x^{\frac{3}{2}}\ln x - \frac{2}{3}x^{\frac{3}{2}}\right] + C = \frac{2}{9}x^{\frac{3}{2}}[3\ln x - 2] + C$

(k) $\int \ln\left(x+\sqrt{1+x^2}\right)dx \quad = x\ln\left(x+\sqrt{1+x^2}\right) - \int x d\left[\ln\left(x+\sqrt{1+x^2}\right)\right]$

$= x\ln\left(x+\sqrt{1+x^2}\right) - \int x\frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{2x}{2\sqrt{1+x^2}}\right)dx = x\ln\left(x+\sqrt{1+x^2}\right) - \int \frac{x}{\sqrt{1+x^2}}dx$

$= x\ln\left(x+\sqrt{1+x^2}\right) - \sqrt{1+x^2} + C$

(l) $\int \left(\sin^{-1}x\right)^2 dx = x\left(\sin^{-1}x\right)^2 - \int x d\left(\left(\sin^{-1}x\right)^2\right) = x\left(\sin^{-1}x\right)^2 - \int x\left[2\left(\sin^{-1}x\right)\right]\frac{dx}{\sqrt{1-x^2}} = x\left(\sin^{-1}x\right)^2 + 2\int\left(\sin^{-1}x\right)d\sqrt{1-x^2}$

$= x\left(\sin^{-1}x\right)^2 + 2\left(\sin^{-1}x\right)\sqrt{1-x^2} - 2\int\sqrt{1-x^2}\frac{dx}{\sqrt{1-x^2}} = x\left(\sin^{-1}x\right)^2 + 2\left(\sin^{-1}x\right)\sqrt{1-x^2} - 2x + C$

(m) $\int_{1/2}^{1} x\ln\left(\frac{1}{x}\right)dx = \frac{1}{2}\int_{x=1/2}^{x=1}\ln\left(\frac{1}{x}\right)dx^2 = \frac{1}{2}\left[x^2\ln\left(\frac{1}{x}\right)\right]_{1/2}^{1} - \int_{1/2}^{1}x^2\left[\frac{1}{1/x}\left(-\frac{1}{x^2}\right)dx\right]$

$= \frac{1}{2}\left[-\frac{1}{4}\ln 2 + \int_{1/2}^{1} x dx\right] \quad = -\frac{1}{8}\ln 2 + \frac{1}{4}x^2\Big|_{1/2}^{1} \quad = \frac{3}{16} - \frac{1}{8}\ln 2$

(o) $\int_0^{\pi/4} e^{2x}\sin^2 x dx = \frac{1}{2}\int_0^{\pi/4} e^{2x}(1-\cos 2x)dx = \frac{1}{2}\int_0^{\pi/4}e^{2x}dx - \frac{1}{4}\int_0^{\pi/4}e^{2x}\cos 2x\, d(2x)$

$= \frac{1}{4}e^{2x}\Big|_0^{\pi/4} - \frac{1}{4}\left[\frac{1}{2}e^{2x}(\sin 2x + \cos 2x)\right]_0^{\pi/4}$, by **1 (h)**

$= \frac{1}{4}\left(e^{\pi/2}-1\right) - \frac{1}{8}\left(e^{\pi/2}-1\right) = \frac{1}{8}\left(e^{\pi/2}-1\right)$

(p) $\int_0^{1/2} x\ln\frac{1+x}{1-x}dx = \frac{1}{2}\int_{x=0}^{x=1/2}\ln\frac{1+x}{1-x}dx^2 = \frac{1}{2}\left[\left[x^2\ln\frac{1+x}{1-x}\right]_0^{1/2} - \int_0^{1/2}x^2\frac{1-x}{1+x}\frac{(1-x)-(1+x)(-1)}{(1-x)^2}dx\right]$

$= \frac{1}{2}\left[\frac{1}{4}\ln 3 - \int_0^{1/2}\frac{2x^2}{(1-x)(1+x)}dx\right] = \frac{1}{8}\ln 3 - \int_0^{1/2}\left[1 + \frac{1/2}{x-1} - \frac{1/2}{x+1}\right]dx$

$= \frac{1}{8}\ln 3 + \left[x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1|\right]_0^{1/2} = \frac{1}{8}(4 - 3\ln 3)$

2. (a) $3\sqrt[3]{\frac{s}{2}} + C$ **(b)** $\frac{4}{3}y^{\frac{3}{2}} - \frac{25}{11}y^{\frac{11}{5}} + \frac{5}{2}y^{\frac{4}{5}} + \frac{1}{3}y^3 + 3y + C$ **(c)** $\frac{1}{2(2-3z)^2} + C$

(d) $\frac{1}{10}(2x+5)^5 + C$ **(e)** $\ln 3$ **(f)** $-\frac{3}{2}\ln|1-3p| + C$ **(g)** $\frac{2}{3}e^{3x+2} + C$

(h) $-\frac{2}{3}e^{-\frac{3}{10}y} + C$ **(i)** $\frac{1}{3\ln 2}2^{3z} + C$ **(j)** $\frac{1}{3}\cos(2-3\theta) + C$ **(k)** $\frac{3\sqrt{3}}{2}$

(l) Let $t = 3\tan\theta$, $dt = 3\sec^2\theta\, d\theta$.

When $t = 3$, $\theta = \pi/4$. When $t = 0$, $\theta = 0$. $\therefore \int_0^3 \frac{dt}{9+t^2} = \int_0^{\pi/4}\frac{3\sec^2\theta d\theta}{9+(3\tan\theta)^2} = \frac{1}{3}\int_0^{\pi/4}d\theta = \frac{\pi}{12}$

(m) Let $x = 4\sin\theta$, $dt = 4\cos\theta\, d\theta$

When $x = 2$, $\theta = \pi/6$. When $x = 0$, $\theta = 0$.

$\int_0^2 \frac{dx}{16-x^2} = \int_0^{\pi/6}\frac{4\cos\theta d\theta}{16-(4\sin\theta)^2} = \frac{1}{4}\int_0^{\pi/6}\sec\theta d\theta = \frac{1}{4}\left[\ln\left|\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right|\right]_0^{\pi/6} = \frac{1}{4}\left[\ln\tan\frac{\pi}{3} - \ln\tan 1\right] = \frac{\ln 3}{8}$

(n) Let $\sqrt{2y-y^2} = yt$ $\quad\therefore 2y - y^2 = y^2t^2$.

$$t^2 = \frac{2-y}{y}, \qquad y = \frac{2}{t^2+1}, \qquad dy = -\frac{4t}{(t^2+1)^2}dt$$

When $y = 1$, $t = 1$; when $y = 0$, $t = \infty$.

$$\int_0^1 \frac{dy}{\sqrt{2y-y^2}} = \int_\infty^1 \frac{-\frac{4t}{(t^2+1)^2}dt}{\left(\frac{2}{t^2+1}\right)t} = -2\int_\infty^1 \frac{dt}{t^2+1} = -2(\tan^{-1} t)_\infty^1 = -2\left(\frac{\pi}{4}-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

2. (o) Let $z = \frac{3}{2}\tan\theta$, $\quad dz = \frac{3}{2}\sec^2\theta d\theta$

$$I = \int \frac{\frac{3}{2}\sec^e\theta d\theta}{3\sec\theta} = \frac{1}{2}\int \sec\theta d\theta = \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C = \frac{1}{2}\ln\left|\sqrt{\left(\frac{2z}{3}\right)^2+1} + \frac{2z}{3}\right| + C = \frac{1}{2}\ln\left|2z + \sqrt{9+4z^2}\right| + C'$$

(p) Let $s = \frac{1}{3}\sec\theta$, $\quad ds = \frac{1}{3}\sec\theta\tan\theta d\theta$

$$\int \frac{ds}{\sqrt{9s^2-1}} = \int \frac{\frac{1}{3}\sec\theta\tan\theta d\theta}{\tan\theta} = \frac{1}{3}\int \sec\theta d\theta = \frac{1}{3}\ln\left|\sec\theta+\tan\theta\right| + C = \frac{1}{3}\ln\left|3s + \sqrt{9s^2-1}\right| + C$$

(q) $I = \frac{1}{2}\ln\left|4t^2+2t-1\right| + C$

(r) $I = \int\left(1 + \frac{4/3}{x-2} - \frac{1/3}{x+1}\right)dx = x + \frac{4}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C$

(s) $I = \frac{1}{3}\int \frac{d(2p^3-3p+2)}{\sqrt{2p^3-3p+2}} = \frac{2}{3}\sqrt{2p^3-3p+2} + C$

(t) $I = \int \frac{dz}{(z+1)^2+9}$ Put $z + 1 = 3 \tan\theta$, $dz = 3 \sec^2\theta$. $\quad I = \frac{1}{3}\tan^{-1}\left(\frac{z+1}{3}\right) + C$

(u) $I = \int_0^1 \frac{(1+x^2)^2 - 2(1+x^2)+1}{1+x^2}dx = \int_0^1\left[(1+x^2) - 2 + \frac{1}{1+x^2}\right]dx = \left[x + \frac{x^3}{3} - 2x + \tan^{-1}x\right]_0^1 = \frac{\pi}{4} - \frac{2}{3}$

(v) $I = \int_0^2 \frac{1+2s+2}{1+2s}ds = \int_0^2\left(1 + \frac{2}{1+2s}\right)ds = [s + \ln|1+2s|]_0^2 = 2 + \ln 5$

(w) $I = \int \frac{dy}{\sqrt{(y+1)^2+3^2}}$ Put $y + 1 = 3 \tan\theta$,

$$I = \int \frac{3\sec^2\theta d\theta}{3\sec\theta} = \int \sec\theta d\theta = \ln\left|\sec\theta+\tan\theta\right| + C = \ln\left|y+1+\sqrt{y^2+2y+10}\right| + C$$

(x) $I = \int \frac{\frac{1}{2}(2u+4)+3}{\sqrt{u^2+4u+5}}du = \frac{1}{2}\int \frac{d(u^2+4u+5)}{\sqrt{u^2+4u+5}} + 3\int \frac{du}{\sqrt{u^2+4u+5}} = \sqrt{u^2+4u+5} + 3\ln\left|u+2+\sqrt{u^2+4u+5}\right| + C$

(y) $\int \frac{p+5}{p^2+4p+5}dp = \int \frac{\frac{1}{2}(2p+4)+3}{p^2+4p+5}dp = \frac{1}{2}\int \frac{d(p^2+4p+5)}{p^2+4p+5} + 3\int \frac{d(p+2)}{(p+2)^2+1} = \frac{1}{2}\ln\left|p^2+4p+5\right| + 3\tan^{-1}(p+2) + C$

(z) $\int \frac{x^n}{1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}}dx = n!\int \frac{\frac{x^n}{n!}}{1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}}dx = n!\int \frac{\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}\right) - \left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^{n-1}}{(n-1)!}\right)}{1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}}dx$

$$= n!\int dx - \int \frac{1+\frac{x}{1!}+\frac{x^2}{2!}+...+\frac{x^{n-1}}{(n-1)!}}{1+\frac{x}{1!}+\frac{x^2}{2!}+...+\frac{x^n}{n!}}dx = n!\left[x - \int \frac{d\left(1+\frac{x}{1!}+\frac{x^2}{2!}+...+\frac{x^n}{n!}\right)}{1+\frac{x}{1!}+\frac{x^2}{2!}+...+\frac{x^n}{n!}}x\right] = n!\left[x - \ln\left(1+\frac{x}{1!}+\frac{x^2}{2!}+...+\frac{x^n}{n!}\right)\right]+C$$

3. (a) $\int \frac{dx}{x^2(x^2+a^2)} = \frac{1}{a^2}\int\left(\frac{1}{x^2}-\frac{1}{x^2+a^2}\right)dx = \frac{1}{a^2}\left(-\frac{1}{x}-\frac{1}{a}\tan^{-1}\frac{x}{a}\right)+C$

(b) $\int_0^2 \frac{dx}{x^3+8} = \int_0^2 \frac{dx}{(x+2)(x^2-2x+4)} = \frac{1}{12}\int_0^2\left(\frac{1}{x+2}-\frac{x-4}{x^2-2x+4}\right)dx = \frac{1}{12}\int_0^2\left(\frac{1}{x+2}-\frac{1}{2}\times\frac{(2x-2)+2}{x^2-2x+4}\right)dx$

$= \frac{1}{12}\left[\int_{x=0}^{x=2}\frac{d(x+2)}{x+2}+\frac{1}{2}\int_{x=0}^{x=2}\frac{d(x^2-2x+4)}{x^2-2x+4}+\int_{x=0}^{x=2}\frac{d(x-1)}{(x-1)^2+3}\right]$

$= \frac{1}{12}\left[\ln|x+2|+\frac{1}{2}\ln|x^2-2x+4|+\frac{3}{\sqrt{3}}\tan^{-1}\frac{x-1}{\sqrt{3}}\right]_0^2 = \frac{1}{72}\left(\ln 64+2\sqrt{3}\pi\right)$

3. (c) $\int \frac{y^3}{(a^2+y^2)^2}dy = \frac{1}{2}\int\frac{y^2 d(a^2+y^2)}{(a^2+y^2)^2} = \frac{1}{2}\int\frac{[(a^2+y^2)-a^2]d(a^2+y^2)}{(a^2+y^2)^2} = \frac{1}{2}\left[\ln|a^2+y^2|+\frac{a^2}{a^2+y^2}\right]+C$

(d) $\int \frac{x-3}{\sqrt{4x^2-4x-3}}dx = \int\frac{\frac{1}{8}(8x-4)-\frac{5}{2}}{\sqrt{4x^2-4x-3}}dx = \frac{1}{4}\int\frac{d(4x^2-4x-3)}{\sqrt{4x^2-4x-3}}-\frac{5}{2}\int\frac{dx}{\sqrt{(2x-1)^2-4}}$

$= \frac{1}{4}\ln|4x^2-4x-3|-\frac{5}{4}\ln\left|2x-1+\sqrt{4x^2-4x-3}\right|+C$

(e) $-\frac{2}{9}\sqrt{18p-9p^2-5}+\frac{7}{3}\sin^{-1}\left[\frac{3(p-1)}{2}\right]+C$

(f) $I = \int\frac{dz}{(z-1)^2(z+1)(z-2)} = \int\left(\frac{-\frac{1}{2}}{(z-1)^2}+\frac{-\frac{1}{4}}{z+1}+\frac{-\frac{1}{4}}{z+1}+\frac{\frac{1}{3}}{z-2}\right)dz = \frac{1}{2(z-1)}-\frac{1}{4}\ln|z-1|-\frac{1}{12}\ln|z+1|+\frac{1}{3}\ln|z-2|+C$

(g) $\int\frac{du}{u^4+3u^2+2} = \int\frac{du}{(u^2+1)(u^2+2)} = \int\left(\frac{1}{u^2+1}-\frac{1}{u^2+2}\right)du = \tan^{-1}u-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+C$

(h) $I = \int\sqrt{10+6x+x^2}\,dx = \int\sqrt{(x+3)^2+1}\,dx$

Let $x+3=\tan\theta$, $dx=\sec^2\theta\,d\theta$. $\therefore I = \int\sec^3\theta\,d\theta = \frac{1}{2}[\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|]+C$, see **1 (b)**

$I = \frac{1}{2}\left[(x+3)\sqrt{10+6x+x^2}+\ln\left|x+3+\sqrt{10+6x+x^2}\right|\right]+C$

(i) $I = \int\sqrt{4v^2-8v-5}\,dv = \int\sqrt{4v^2-8v-5}\,dv = \int\sqrt{(2v-2)^2-9}\,dv$

Let $2v-2 = 3\sec\theta$, $2\,dv = 3\sec\theta\tan\theta\,d\theta$.

$I = \frac{9}{2}\int\sec\theta\tan^2\theta\,d\theta = \frac{9}{2}\int\sec\theta(\sec^2\theta-1)d\theta = \frac{9}{2}\left[\frac{1}{2}(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|)-\ln|\sec\theta+\tan\theta|\right]+C$

$= \frac{9}{4}(\sec\theta\tan\theta-\ln|\sec\theta+\tan\theta|)+C = \frac{v-1}{2}\sqrt{4v^2-8v-5}-\frac{9}{4}\ln\left|2v-2+\sqrt{4v^2-8v-5}\right|+C$

(j) $\int_0^{2\pi}\sin 4\theta\,d\theta = \int_0^{2\pi}2\sin 2\theta\cos 2\theta\,d\theta = \int_0^{2\pi}\sin 2\theta\,d(\sin 2\theta) = \left[\frac{\sin^2 2\theta}{2}\right]_0^{2\pi} = 0$

(k) $\frac{1}{2}$ **(l)** $\frac{2}{3}\sin^3\frac{x}{2}+C$

(m) **(Method 1)** Let $t=\tan\frac{u}{2}$, $du=\frac{2dt}{1+t^2}$, $\sin u=\frac{2t}{1+t^2}$, $\cos u=\frac{1-t^2}{1+t^2}$

$$I=\ldots=\int\frac{dt}{2+3t-2t^2}=-\int\frac{dt}{(2t+1)(t-2)}=-\frac{1}{5}\int\left(\frac{1}{t-2}-\frac{2}{2t+1}\right)dt=-\frac{1}{5}[\ln|t-2|-\ln|2t+1|]+C$$

(Method 2) $3\sin u+4\cos u=5\cos(u-\alpha)$, $\tan\alpha=\frac{3}{4}$

$$I=\int\frac{du}{\cos(u-\alpha)}=\int\sec(u-\alpha)du=\ln|\sec(u-\alpha)+\tan(u-\alpha)|+C,\quad \text{where } \alpha=\tan^{-1}\left(\frac{3}{4}\right).$$

3. (n) $$\int\frac{dt}{(5\sin t-12\cos t)^2}=\int\frac{dt}{\cos^2 t(5\tan t-12)^2}=\int\frac{\sec^2 t\,dt}{(5\tan t-12)^2}=\frac{1}{5}\int\frac{d(5\tan t-12)}{(5\tan t-12)^2}=-\frac{1}{5(5\tan t-12)}+C$$

(o) $$\int_0^{\pi/2}\frac{d\theta}{4+5\cos^2\theta}=\int_0^{\pi/2}\frac{\sec^2\theta\,d\theta}{4\sec^2\theta+5}=\int_0^{\pi/2}\frac{d(\tan\theta)}{4(\tan^2\theta+1)+5}=\int_0^{\pi/2}\frac{d(\tan\theta)}{4\tan^2\theta+9}=\frac{1}{6}\tan^{-1}\left(\frac{2\tan\theta}{3}\right)_0^{\pi/2}=\frac{\pi}{12}$$

(p) $I=\int\frac{dv}{4+5\cos v}$ Let $t=\tan\frac{v}{2}$, $dv=\frac{2dt}{1+t^2}$, $\sin v=\frac{2t}{1+t^2}$, $\cos v=\frac{1-t^2}{1+t^2}$

$$I=\int\frac{2dt}{9-t^2}=-\int\frac{2dt}{(t-3)(t+3)}=\frac{1}{3}\int\left(\frac{1}{t+3}-\frac{1}{t+3}\right)dt=\frac{1}{3}\ln\left|\frac{t+3}{t-3}\right|+C$$

(q) $$\int_0^{\pi/3}\sin^2 2\theta\,d\theta=\int_0^{\pi/3}\frac{1-\cos4\theta}{2}d\theta=\left[\frac{1}{2}\theta-\frac{1}{8}\sin4\theta\right]_0^{\pi/3}=\frac{1}{2}\left[\frac{\pi}{3}+\frac{\sqrt{3}}{8}\right]$$

(r) $$\int\cos2n\theta\sin3n\theta\;d\theta=\frac{1}{2}\int(\sin5n\theta+\sin n\theta)d\theta=-\frac{1}{2n}\left(\frac{1}{5}\cos5n\theta+\cos n\theta\right)+C$$

(s) $$\int\cos2nt\cos5nt\;dt=\frac{1}{2}\int(\cos7nt+\cos3nt)dt=\frac{1}{2n}\left(\frac{1}{7}\sin7nt+\frac{1}{3}\sin3nt\right)+C$$

(t) $$\int\sin5\phi\sin7\phi\,d\phi=\frac{1}{2}\int(\cos12\phi-\cos2\phi)\,d\phi=\frac{1}{4}\left(\sin2\phi-\frac{1}{6}\sin12\phi\right)+C$$

(u) $$\int_0^{\pi/2}\sin^5 x\,dx=-\int_{x=0}^{x=\pi/2}\sin^4 x\,d(\cos x)=-\int_{x=0}^{x=\pi/2}(1-\cos^2 x)^2\,d(\cos x)=-\int_{x=0}^{x=\pi/2}(1-2\cos^2 x+\cos^4 x)\,d(\cos x)$$

$$=-\left[\cos x-\frac{2}{3}\cos^3 x+\frac{1}{5}\cos^5 x\right]_0^{\pi/2}=-\left[-1+\frac{2}{3}-\frac{1}{5}\right]_0^{\pi/2}=\frac{8}{15}$$

(v) $$\int_0^{\pi/2}\sin^3 x\cos^4 x\,dx=-\int_{x=0}^{x=\pi/2}(1-\cos^2 x)\cos^4 x\,d(\cos x)=-\left[\frac{\cos^5 x}{5}-\frac{\cos^7 x}{7}\right]_0^{\pi/2}=\frac{2}{35}$$

(w) $$\int\sin^6 x\,dx=\int\left[\frac{1}{2}(1-\cos2x)\right]^3dx=\frac{1}{8}\int(1-3\cos2x+3\cos^2 2x-\cos^3 2x)dx$$

$$=\frac{1}{8}\int\left[1-3\cos2x+3\left(\frac{1+\cos4x}{2}\right)\right]dx+\frac{1}{16}\int(1-\sin^2 2t)d(\sin2t)=\frac{1}{16}\left[5t-2\sin2t+\frac{3}{4}\sin4t-\frac{1}{3}\sin^3 2t\right]+C$$

Complex number method

$$\sin^6 x=\left[\frac{1}{2i}(e^{ix}-e^{-ix})\right]^6=-\frac{1}{64}(e^{6ix}-6e^{4ix}+15e^{2ix}-20-15e^{-2ix}-6e^{4ix}+e^{-6ix})$$

$$=-\frac{1}{32}\left[\frac{e^{6ix}+e^{-6ix}}{2}-6\frac{e^{4ix}+e^{4ix}}{2}+15\frac{e^{2ix}+e^{-2ix}}{2}-20\right]=-\frac{1}{32}[\cos6x-6\cos4x+15\cos2x-20]$$

$$\int\sin^6 x dx=-\frac{1}{32}\left[\frac{\sin6x}{6}-6\frac{\sin4x}{4}+15\frac{\sin2x}{2}-20\right]+C=\frac{5x}{16}-\frac{15}{64}\sin2x+\frac{3}{64}\sin4x-\frac{1}{192}\sin6x+C$$

(x) $$\sin^2\theta\cos^4\theta=\left[\frac{1}{2i}(e^{i\theta}-e^{-i\theta})\right]^2\left[\frac{1}{2}(e^{i\theta}+e^{-i\theta})\right]^4=-\frac{1}{64}(e^{2i\theta}-2+e^{-2i\theta})(e^{4i\theta}+4e^{2i\theta}+6+4e^{-2i\theta}+e^{-4i\theta})$$

$$= -\frac{1}{32}\left[\frac{e^{6i\theta}+e^{-6i\theta}}{2} + 2\times\frac{e^{4i\theta}+e^{-4i\theta}}{2} - \frac{e^{i\theta}+e^{-i\theta}}{2} + 2\right] = -\frac{1}{32}[\cos 6\theta + 2\cos 4\theta - \cos 2\theta + 2]$$

$$\int \sin^2\theta\cos^4\theta\, d\theta = -\frac{1}{32}\left[\frac{\sin 6\theta}{6} + 2\times\frac{\sin 4\theta}{4} - \frac{\sin 2\theta}{2} + 2\theta\right] + C = \frac{\theta}{16} + \frac{1}{64}\sin 2\theta - \frac{1}{64}\sin 4\theta - \frac{1}{192}\sin 6\theta + C$$

4. (a) $I = \int_{-\pi/2}^{\pi/2} \sin^5\phi\cos^3\phi\, d\phi$. Put $\phi = -\theta$, $d\phi = -d\theta$, $I = \int_{+\pi/2}^{-\pi/2} \sin^5(-\theta)\cos^3(-\theta)(-d\theta) = -I$

$\therefore$ I = 0.

(b) $\int_0^{\pi/4} \tan^4 x\, dx = \int_0^{\pi/4} \tan^2 x(\sec^2 x - 1)\, dx = \int_0^{\pi/4} (\tan^2 x\sec^2 x - \tan^2 x)dx = \int_0^{\pi/4} (\tan^2 x\sec^2 x - \sec^2 x + 1)dx$

$$= \int_0^{\pi/4} (\tan^2 x - 1)d(\tan x) - \int_0^{\pi/4} dx = \left[\frac{1}{3}\tan^3 x - \tan x\right]_0^{\pi/4} - [x]_0^{\pi/4} = \frac{\pi}{4} - \frac{2}{3}$$

(c) $-\cot\theta - \dfrac{\cot^3\theta}{3} + C$

(d) Put $z = \sqrt{y^2+1} - y$, $z + y = \sqrt{y^2+1}$, $z^2 + 2yz + y^2 = y^2 + 1$, $y = \dfrac{1-z^2}{2z}$, $dy = -\dfrac{z^2+1}{2z^2}dz$

$$\int \frac{dy}{\sqrt{y^2+1} - y} = \int \frac{1}{z}\left(-\frac{z^2+1}{2z^2}dz\right) = -\frac{1}{2}\int\left(\frac{1}{z} + \frac{1}{z^3}\right)dz = -\frac{1}{2}\left(\ln|z| - 2\frac{1}{z^2}\right) + C$$

$$= -\frac{1}{2}\left(\ln\left|\sqrt{y^2+1} - y\right| - 2\frac{1}{\left(\sqrt{y^2+1} - y\right)^2}\right) + C = \frac{1}{2}\left[y^2 + y\sqrt{y^2+1} + \ln\left|\sqrt{y^2+1} + y\right|\right] + C$$

(e) $I = \int\sqrt{\dfrac{3-x}{x-1}}\, dx$. Let x = 3 cos $^2\theta$ + sin^2 θ = 2cos $^2\theta$ + 1= 3 – 2 sin^2 θ, dx = -4 sin θ cos θ dθ

$$I = \int\sqrt{\frac{3 - 3\cos^2\theta - \sin^2\theta}{2\cos^2\theta}}(-4\sin\theta\cos\theta d\theta) = -4\int\sin^2\theta d\theta = 4\int\frac{1-\cos 2\theta}{2}d\theta = 2\int(1-\cos 2\theta)d\theta$$

$$= 2\theta - \sin 2\theta + C = 2\theta - 2\sin\theta\cos\theta + C = 2\sin^{-1}\sqrt{\frac{3-x}{2}} - 2\sqrt{\frac{3-x}{2}}\sqrt{\frac{x-1}{2}} + C = 2\sin^{-1}\sqrt{\frac{3-x}{2}} - \sqrt{3-x}\sqrt{x-1} + C$$

(f) $\dfrac{1}{3}(a^2 - z^2)^{-3/2} + C$

(g) Put p = a tan θ, $\int\dfrac{dp}{(a^2+p^2)^{3/2}} = \dfrac{p}{a^2\sqrt{a^2+p^2}} + C$

(h) Let v = a sin θ , $\int_0^a (a^2 - v^2)^{3/2} dv = a^4\int_0^{\pi/2}\cos^4\theta d\theta = a^4\int_0^{\pi/2}\left(\dfrac{1+\cos 2\theta}{2}\right)^2 d\theta = ... = \dfrac{3\pi a^4}{16}$

(i) Let y = 1 – 3x , $\int\dfrac{x^2\, dx}{\sqrt{1-3x}} = -\dfrac{2}{27}\left[\sqrt{1-3x} - \dfrac{2}{3}(1-3x)^{3/2} + \dfrac{1}{5}(1-3x)^{5/2}\right] + C$

(j) $I = \int_0^{2a}\dfrac{x}{\sqrt{2ax - x^2}}dx = \int_0^{2a}\dfrac{x}{\sqrt{a^2 - (a-x)^2}}dx$, Let a – x = a sin θ, – dx = a cos θ dθ , x = a (1 – sin θ)

$$I = \int_{\pi/2}^{-\pi/2}\frac{a(1-\sin\theta)}{a\cos\theta}(-a\cos\theta d\theta) = a\int_{-\pi/2}^{\pi/2}(1-\sin\theta)d\theta = a[\theta + \cos\theta]_{-\pi/2}^{\pi/2} = \pi a$$

(k) $I = \int_a^b\dfrac{dy}{\sqrt{(y-a)(b-y)}}$, Let y = a cos^2 θ + b sin^2 θ , dy = 2 (b – a) sin θ cos θ

y – a = (b – a) sin^2 θ , b – y = (b – a) cos^2 θ

$$\therefore I = \int_0^{\pi/2}\frac{2(b-a)\sin\theta\cos\theta}{(b-a)\sin\theta\cos\theta}d\theta = 2\int_0^{\pi/2}d\theta = 2\left(\frac{\pi}{2}\right) = \pi$$

(l) $\frac{2}{35}$

(m) $t=\tan\frac{\theta}{2}$, $\quad d\theta=\frac{2dt}{1+t^2}$, $\quad \cos\theta=\frac{1-t^2}{1+t^2}$

$$\int\frac{\cos\theta}{1+2\cos\theta}d\theta=..=2\int\frac{1-t^2}{(3-t^2)(1+t^2)}dt=\int\left(\frac{1}{1+t^2}-\frac{1}{3-t^2}\right)dt=\tan^{-1}t-\frac{1}{2\sqrt{3}}\ln\frac{t+\sqrt{3}}{t-\sqrt{3}}+C=\theta-\frac{1}{2\sqrt{3}}\ln\frac{\tan\frac{\theta}{2}+\sqrt{3}}{\tan\frac{\theta}{2}-\sqrt{3}}+C$$

(n) Let $x=\sec y$, $dx=\sec y\tan y\,dy$ $\quad \therefore\int\frac{dx}{x\sqrt{x^2-1}}=\int\frac{\sec y\tan y dy}{\sec y\tan y}=\int dy=y+C=\sec^{-1}y+C$

(o) $\int\sin^2 2\theta d\theta=\int\frac{1-\cos 4\theta}{2}d\theta=\frac{1}{2}\left(\theta-\frac{\sin 4\theta}{4}\right)+C$

(p) $y=\sin^{-1}\theta$, $\sin y=x$, $\cos y\,dy=dx$

$$\int\sin^{-1}x\,dx=\int y(\cos y dy)=\int yd(\sin y)=y\sin y-\int\sin y dy=y\sin y+\cos y+C=x\sin^{-1}x+\sqrt{1-x^2}+C$$

(q) $\int v^3\ln v\,dv=\frac{1}{4}\int\ln v d(v^4)=\frac{1}{4}\left[v^4\ln v-\int v^4\frac{dv}{v}\right]=\frac{1}{4}\left(v^4\ln v-\frac{v^4}{4}\right)+C$

(r) $I=\int e^{-2\theta}\sin 3\theta d\theta=-\frac{1}{2}\int\sin 3\theta d(e^{-2\theta})=-\frac{1}{2}\left(e^{-2\theta}\sin 3\theta-3\int e^{-2\theta}\cos 3\theta d\theta\right)=-\frac{1}{2}e^{-2\theta}\sin 3\theta+\frac{3}{2}\times\left(-\frac{1}{2}\right)\int\cos 3\theta d(e^{-2\theta})$

$=-\frac{1}{2}e^{-2\theta}\sin 3\theta-\frac{3}{4}\left(e^{-2\theta}\cos 3\theta+3\int e^{-2\theta}\sin 3\theta d\theta\right)=-\frac{1}{2}e^{-2\theta}\sin 3\theta-\frac{3}{4}\left(e^{-2\theta}\cos 3\theta+3I\right)$

$\therefore I=-\frac{1}{13}(3\cos 3\theta+2\sin 3\theta)+C$

(s) $\int x\sqrt{1+x}\,dx=\int[(1+x)-1]\sqrt{1+x}\,dx=\int(1+x)^{3/2}dx-\int(1+x)dx=\frac{2}{5}(1+x)^{5/2}+\frac{2}{3}(1+x)^{3/2}+C$

(t) $\int u^2e^{-\frac{1}{2}u}du=-2\int u^2d(e^{-\frac{1}{2}u})=-2\left[u^2e^{-\frac{1}{2}u}-\int 2ue^{-\frac{1}{2}u}du\right]=-2\left[u^2e^{-\frac{1}{2}u}+4\int ud(e^{-\frac{1}{2}u})\right]=...=-2e^{-\frac{1}{2}u}(u^2+4u+8)+C$

(u) $\int\frac{xe^x dx}{(1+x)^2}=-\int xe^xd\left(\frac{1}{1+x}\right)=-\left[\frac{xe^x}{1+x}-\int\frac{1}{1+x}d(xe^x)\right]=-\frac{xe^x}{1+x}+\int\frac{e^x+xe^x}{1+x}dx=-\frac{xe^x}{1+x}+\int e^xdx=...=\frac{e^x}{1+x}+C$

(v) $\int_0^{\pi/2}\sin^3\theta(\sin^3\theta+\cos^3\theta)d\theta=\int_0^{\pi/2}\sin^6\theta d\theta+\int_0^{\pi/2}\sin^3\theta\cos^3\theta d\theta$

$=\frac{1}{16}\left[5\theta-2\sin 2\theta+\frac{3}{4}\sin 4\theta-\frac{1}{3}\sin^3 2\theta\right]_0^{\pi/2}+\int_0^{\pi/2}\sin^3\theta\cos^2\theta d(\sin\theta)$, by 3. (w)

$=\frac{5\pi}{32}+\int_0^{\pi/2}\sin^3\theta(1-\sin^2\theta)d(\sin\theta)=\frac{5\pi}{32}+\frac{1}{12}$

5. (a) $x\sec^{-1}x-\ln\left|x+\sqrt{x^2-1}\right|+C$ **(b)** $\frac{1}{2}\left[x^2\ln(x+1)-\frac{x^2}{2}+x-\ln(x+1)\right]+C$ **(c)** $\frac{1}{2}\left[\theta^2\sin 2\theta+\theta\cos 2\theta-\frac{\sin 2\theta}{2}\right]+C$

(d) $-\ln|\sin\theta-1|-\theta-\frac{\sin^2\theta}{2}+C$ **(e)** $\frac{7}{15}$ **(f)** Let $x=\frac{\tan\theta}{2}$, $\quad I=\ln\left|\frac{\sqrt{1+4x^2}-1}{2x}\right|+C$

(g) $\int_{\pi/2}^{\pi/4}\cot^3\theta d\theta=\int_{\pi/2}^{\pi/4}(\csc^2\theta-1)\cot\theta d\theta=\int_{\pi/2}^{\pi/4}\csc^2\theta\cot\theta d\theta-\int_{\pi/2}^{\pi/4}\cot\theta d\theta=\int_{\pi/2}^{\pi/4}\cot\theta d(\cot\theta)-\int_{\pi/2}^{\pi/4}\cot\theta d\theta$

$=\left[-\frac{\cot^2\theta}{2}-\ln(\sin\theta)\right]_{\pi/2}^{\pi/4}=-\frac{1}{2}-\ln\frac{1}{\sqrt{2}}=\frac{1}{2}(\ln 2-1)$

(h) Let $x=\cos\theta$, $dx=-\sin\theta\,d\theta$,

$$\int\frac{dx}{(1+x)\sqrt{1-x^2}}=-\int\frac{\sin\theta d\theta}{(1+\cos\theta)\sin\theta}=\int\frac{d\theta}{2\cos^2\frac{\theta}{2}}=\int\sec^2\frac{\theta}{2}d\left(\frac{\theta}{2}\right)\quad=-\tan\left(\frac{\theta}{2}\right)+C=-\frac{1-\cos\theta}{\sin\theta}+C=-\frac{1-x}{\sqrt{1-x^2}}+C$$

5. (i) $$\int\frac{x+\sin x}{1+\cos x}dx=\int\frac{x+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}dx=\int\left(\frac{1}{2}x\sec^2\frac{x}{2}+\tan\frac{x}{2}\right)dx=\int\left[xd\left(\tan\frac{x}{2}\right)+\tan\frac{x}{2}dx\right]=\int d\left(x\tan\frac{x}{2}\right)=x\tan\frac{x}{2}+C$$

(j) Let $x=a\sin^2\theta+b\cos^2\theta$, Then $\sqrt{b-x}=\sqrt{b-a}\sin\theta$, $\sqrt{x-a}=\sqrt{b-a}\cos\theta$, $dx = -2(b-a)\sin\theta\cos\theta\, d\theta$.

$$\int_a^b\sqrt{\frac{b-x}{x-a}}dx=\int_{\pi/2}^{0}\frac{\sqrt{b-a}\sin\theta}{\sqrt{b-a}\cos\theta}[-2(b-a)\sin\theta\cos\theta d\theta]=2(b-a)\int_0^{\pi/2}\sin^2\theta d\theta=\ldots=\frac{b-a}{2}\pi$$

(k) $x=\sqrt{\frac{\beta}{\alpha}}\tan\theta$, $dx=\sqrt{\frac{\beta}{\alpha}}\sec^2\theta d\theta$

$$\int\frac{dx}{x^2\sqrt{\alpha x^2+\beta}}=\int\frac{\alpha}{\beta\tan\theta\left(\sqrt{\beta}\sec\theta\right)}\sqrt{\frac{\beta}{\alpha}}\sec^2\theta d\theta=\frac{\sqrt{\alpha}}{\beta}\int\csc\theta\cot\theta d\theta=-\frac{\sqrt{\alpha}}{\beta}\cot\theta+C=-\frac{\sqrt{\alpha x^2+\beta}}{\beta}+C$$

(l) $$\int\frac{dx}{(x+1)(x+2)^2(x+3)^3}=\int\left(\frac{1/8}{x+1}+\frac{2}{x+2}-\frac{1}{(x+2)^2}-\frac{17/8}{x+3}-\frac{5/4}{(x+3)^2}-\frac{1/2}{(x+3)^3}\right)dx$$

$$=\frac{1}{8}\ln(x+1)+2\ln(x+2)+\frac{1}{x+2}-\frac{17}{8}\ln(x+3)+\frac{5}{4(x+3)}+\frac{1}{4(x+3)^2}+C$$

(m) $$\int\frac{x^2+5x+4}{x^4+5x^2+4}dx=\int\frac{x^2+5x+4}{\left(x^2+1\right)\left(x^2+4\right)}dx=\int\left(\frac{\frac{5}{3}x+1}{x^2+1}-\frac{\frac{5}{3}x}{x^2+4}\right)dx=\int\frac{dx}{x^2+1}+\frac{5}{6}\int\frac{2xdx}{x^2+1}-\frac{5}{6}\int\frac{2xdx}{x^2+4}$$

$$=\tan^{-1}x+\frac{5}{6}\ln\left(x^2+1\right)-\frac{5}{6}\ln\left(x^2+4\right)+C$$

(n) $$\int\frac{dx}{\left(x^2-4x+4\right)\left(x^2-4x+5\right)}=\int\left(\frac{1}{x^2-4x+4}-\frac{1}{x^2-4x+5}\right)dx=\int\frac{dx}{(x-2)^2}-\int\frac{dx}{(x-2)^2+1^2}=-\frac{1}{x-2}-\tan^{-1}(x-2)+C$$

(o) Let $y=\frac{x+1}{x-1}$, $dy=-\frac{2dx}{(x-1)^2}$,

$$\int\frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}=-\frac{1}{2}\int\sqrt[3]{\left(\frac{x-1}{x+1}\right)^2}\left[-\frac{2dx}{(x-1)^2}\right]=-\frac{1}{2}\int\sqrt[3]{\frac{1}{y^2}}dy=-\frac{3}{2}\sqrt[3]{y}+C=-\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}}+C$$

(p) Let $y=1+\sqrt[4]{x}$, $\sqrt{x}=(y-1)^2$, $x=(y-1)^4$, $dx=4(y-1)^3dy$

$$\int\frac{dx}{\sqrt{x}\left(1+\sqrt[4]{x}\right)^3}=\int\frac{4(y-1)^3dy}{(y-1)^2y^3}=4\int\frac{y-1}{y^3}dy=4\left(-\frac{1}{y}+\frac{1}{2y^2}\right)+C=\frac{2}{\left(1+\sqrt[4]{x}\right)^2}-\frac{4}{1+\sqrt[4]{x}}+C$$

(q) Let $t=\sqrt[4]{\frac{a-x}{x}}$ $t^4=\frac{a-x}{x}$, $x=\frac{a}{t^4+1}$, $dx=-\frac{4at^3dt}{\left(t^4+1\right)^2}$

$$\int\frac{x\,dx}{\sqrt[4]{x^3(a-x)}}=\int\sqrt[4]{\frac{x}{a-x}}dx=\int\frac{1}{t}\left(-\frac{4at^3dt}{\left(t^4+1\right)^2}\right)=-4a\int\frac{t^2dt}{\left(t^4+1\right)^2}=-a\int\frac{1}{t}\left(\frac{4t^3dt}{\left(t^4+1\right)^2}\right)=a\int\frac{1}{t}d\left(\frac{t^4}{t^4+1}\right)$$

$$=-a\left[\frac{1}{t}\times\frac{t^4}{t^4+1}-\int\frac{t^4}{t^4+1}\left(-\frac{1}{t^2}\right)dt\right]=-\frac{at^3}{t^4+1}-a\int\frac{t^2}{t^4+1}dt=-\frac{at^3}{t^4+1}-a\int\frac{t^2}{\left(1-\sqrt{2}t+t^2\right)\left(1+\sqrt{2}t+t^2\right)}dt$$

$$= -\frac{at^3}{t^4+1} - a\int \frac{\frac{1}{4\sqrt{2}}\left[\left(1+\sqrt{2}t+t^2\right)\left(2t-\sqrt{2}\right)-\left(1-\sqrt{2}t+t^2\right)\left(2t+\sqrt{2}\right)+\frac{1}{2}\left(t^2+1\right)\right]}{\left(1-\sqrt{2}t+t^2\right)\left(1+\sqrt{2}t+t^2\right)}dt$$

$$= -\frac{at^3}{t^4+1} - \frac{a}{4\sqrt{2}}\int \frac{2t+\sqrt{2}}{1+\sqrt{2}t+t^2}dt - \frac{a}{4\sqrt{2}}\int \frac{2t-\sqrt{2}}{1-\sqrt{2}t+t^2}dt - \frac{a}{2}\int \frac{t^2+1}{t^4+1}dt$$

$$= -\frac{at^3}{t^4+1} - \frac{a}{4\sqrt{2}}\int \frac{d\left(1+\sqrt{2}t+t^2\right)}{1+\sqrt{2}t+t^2} - \frac{a}{4\sqrt{2}}\int \frac{d\left(1-\sqrt{2}t+t^2\right)}{1-\sqrt{2}t+t^2} - \frac{a}{2\sqrt{2}}\int \frac{d\left(\frac{1-t^2}{\sqrt{2}t}\right)}{\left(\frac{1-t^2}{\sqrt{2}t}\right)^2+1}$$

$$= -\frac{at^3}{t^4+1} - \frac{a}{4\sqrt{2}}\ln\left|1+\sqrt{2}t+t^2\right| - \frac{a}{4\sqrt{2}}\ln\left|1-\sqrt{2}t+t^2\right| + \frac{a}{2\sqrt{2}}\tan^{-1}\left(\frac{1-t^2}{\sqrt{2}t}\right) + C\text{, where } t=\sqrt[4]{\frac{a-x}{x}}$$

5. (r) $\sqrt{ax^2+bx+c} = \sqrt{a\left[\left(x+\frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2\right]}$, Put $x+\frac{b}{2a} = k\sec\theta$, where $k = \frac{\sqrt{b^2-4ac}}{2a}$.

$dx = k\sec\theta\tan\theta\, d\theta$, $\sqrt{ax^2+bx+c} = \sqrt{a}\,k\tan\theta$

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \int \frac{k\sec\theta\tan\theta d\theta}{\sqrt{a}k\tan\theta} = \frac{1}{\sqrt{a}}\int \sec\theta d\theta = \frac{1}{\sqrt{a}}\ln|\sec\theta+\tan\theta| + C = \frac{1}{\sqrt{a}}\ln\left|\left(x+\frac{b}{2a}\right)+\sqrt{\frac{ax^2+bx+c}{a}}\right| + C'$$

(s) $I = \int \sqrt{2+x-x^2}\,dx = \int \sqrt{\left(\frac{3}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}\,dx$. Let $x-\frac{1}{2} = \frac{3}{2}\sin\theta$, $dx = \frac{3}{2}\cos\theta d\theta$

$$I = \frac{9}{4}\int \cos^2\theta d\theta = \frac{9}{8}\int (1+\cos 2\theta)d\theta = \frac{9}{8}\left[\frac{\sin 2\theta}{2}+\theta\right] + C = \ldots = \frac{2x-1}{4}\sqrt{2+x-x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x-1}{3}\right) + C$$

(t) $I = \int x\sqrt{x^4+2x^2-1}\,dx = \int x\sqrt{\left(x^2+1\right)^2-\left(\sqrt{2}\right)^2}\,dx$

Let $x^2+1 = \sqrt{2}\sec\theta$, $2xdx = \sqrt{2}\sec\theta\tan\theta d\theta$

$$I = \int \sec\theta\tan^2\theta d\theta = \int \sec\theta\left(\sec^2\theta-1\right)d\theta = \int\left(\sec^3\theta - \sec\theta\right)d\theta = \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right) - \ln|\sec\theta+\tan\theta| + C\text{, by 1(b).}$$

$$= \frac{1}{2}\left(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\right) + C = \frac{1}{4}\left(x^2+1\right)\sqrt{x^4+2x^2-1} - 2\ln\left|x^2+1+\sqrt{x^4+2x^2-1}\right| + C$$

(u) Let $x^2 = \tan\theta$, $2xdx = \sec^2\theta\, d\theta$

$$\int \frac{x^2+1}{x\sqrt{x^4+1}}dx = \frac{1}{2}\int \frac{\tan\theta+1}{\tan\theta\sqrt{\tan^2\theta+1}}\sec^2\theta d\theta = \frac{1}{2}\int \frac{\tan\theta+1}{\tan\theta}\sec\theta d\theta = \frac{1}{2}\int(\sec\theta+\csc\theta)d\theta$$

$$= \frac{1}{2}\left(\ln|\sec\theta+\tan\theta| + \ln|\csc\theta-\cot\theta|\right) + C = \frac{1}{2}\left(\ln\left|x^4+x^2\sqrt{x^4+1}\right| - \ln\left|1+\sqrt{x^4+1}\right|\right) + C =$$

(v) Let $x-\frac{1}{2} = \frac{\sqrt{5}}{2}\sin\theta$, $dx = \frac{\sqrt{5}}{2}\cos\theta d\theta$

$$\int\frac{x^2\,dx}{\sqrt{1+x-x^2}}=\int\frac{x^2dx}{\sqrt{\left(\frac{\sqrt5}{2}\right)^2-\left(x-\frac12\right)^2}}=\int\frac{\left(\frac{\sqrt5}{2}\sin\theta+\frac12\right)^2}{\frac{\sqrt5}{2}\cos\theta}\frac{\sqrt5}{2}\cos\theta d\theta=\int\left(\frac54\sin^2\theta+\frac{\sqrt5}{2}\sin\theta+\frac14\right)d\theta$$

$$=\int\left[\frac54\left(\frac{1-\cos2\theta}{2}\right)+\frac{\sqrt5}{2}\sin\theta+\frac14\right]d\theta=\frac54\theta-\frac{5\sin2\theta}{16}-\frac{\sqrt5}{2}\cos\theta+\frac14\theta+C=\frac78\theta-\frac{5\sin2\theta}{16}-\frac{\sqrt5}{2}\cos\theta+C$$

$$=\frac78\sin^{-1}\left(\frac{2x-1}{\sqrt5}\right)-\frac58\left(\frac{2x-1}{\sqrt5}\right)\left(\frac{2\sqrt{1+x-x^2}}{\sqrt5}\right)-\frac{\sqrt5}{2}\left(\frac{2\sqrt{1+x-x^2}}{\sqrt5}\right)+C=\frac78\sin^{-1}\left(\frac{2x-1}{\sqrt5}\right)-\frac{(2x+3)}{4}\sqrt{1+x-x^2}+C$$

6. (a) $$I=\int\frac{x\,dx}{(1+x)\sqrt{1-x-x^2}}=\int\frac{(1+x)-1}{(1+x)\sqrt{1-x-x^2}}dx=\int\frac{dx}{\sqrt{1-x-x^2}}-\int\frac{dx}{(1+x)\sqrt{1-x-x^2}}=I_1-I_2$$

$$I_1=\int\frac{2dx}{\sqrt{\left(\sqrt5\right)^2-(2x+1)^2}}=\sin^{-1}\left(\frac{2x+1}{\sqrt5}\right)+C_1,\quad \text{by } 2x+1=\sqrt5\sin\theta$$

For I_2, let $1+x=\frac1t$, $\quad dx=-\frac{1}{t^2}dt$, $\quad x=\frac{1-t}{t}$,

$$I_2=\int\frac{-\frac{1}{t^2}dt}{\frac1t\sqrt{1-\frac{1-t}{t}-\left(\frac{1-t}{t}\right)^2}}=-\int\frac{dt}{\sqrt{t^2+t-1}}=-\int\frac{dt}{\sqrt{\left(t+\frac12\right)^2-\left(\frac{\sqrt5}{2}\right)^2}}$$

Put $t+2=\frac{\sqrt5}{2}\sec\phi$, $dt=\frac{\sqrt5}{2}\sec\phi\tan\phi d\phi$

$$I_2=-\int\sec\phi\,d\phi=-\ln\left|\sec\phi+\tan\phi\right|+C_2=-\ln\left|\frac{2t+1}{\sqrt5}+\frac{2}{\sqrt5}\sqrt{t^2+t-1}\right|+C_2=-\ln\left|2t+1+2\sqrt{t^2+t-1}\right|+C_2'$$

$$=-\ln\left|2\left(\frac{1}{1+x}+1+\frac{2}{1+x}\sqrt{1-x-x^2}\right)\right|+C_2'=-\ln\left|\frac{x+3+2\sqrt{1-x-x^2}}{1+x}\right|+C_2'$$

$$\therefore I=\sin^{-1}\left(\frac{2x+1}{\sqrt5}\right)+\ln\left|\frac{x+3+2\sqrt{1-x-x^2}}{1+x}\right|+C$$

6. (b) Let $y=x+\sqrt{x^2+x-1}$, $\quad (y-x)^2=x^2+x-1$, $\quad x=\frac{y^2-1}{2y+1}$, $\quad dx=\frac{2\left(y^2+y+1\right)}{(2y+1)^2}dy$

$$\int\frac{dx}{x+\sqrt{x^2+x+1}}=\int\frac{2\left(y^2+y+1\right)}{y(2y+1)^2}dy=\int\left[\frac2y-\frac{3}{2y+1}-\frac{3}{(2y+1)^2}\right]dy=2\ln|y|-\frac32\ln|2y+1|+\frac{3}{2(2y+1)}+C$$

$$=2\ln\left|x+\sqrt{x^2+x-1}\right|-\frac32\ln\left|2x+1+2\sqrt{x^2+x-1}\right|+\frac{2x+1-2\sqrt{x^2+x-1}}{2x(x+1)}+C$$

(c) $I=\int\frac{dx}{x-\sqrt{-3+4x-x^2}}=\int\frac{dx}{x-\sqrt{(x-3)(1-x)}}$. Let $\sqrt{(x-3)(1-x)}=t(x-3)$, then $1-x=t^2(x-3)$.

$$\therefore x=\frac{3t^2+1}{t^2+1}=3-\frac{2}{t^2+1},\qquad dx=\frac{4tdt}{\left(t^2+1\right)^2},\quad \sqrt{(x-3)(1-x)}=t(x-3)=t\left(-\frac{2}{t^2+1}\right)=-\frac{2t}{t^2+1}$$

$$\therefore I=\ldots=\int\frac{4tdt}{\left(t^2+1\right)\left(3t^2+2t+1\right)}=\int\left(\frac{-t+1}{t^2+1}+\frac{3t-1}{3t^2+2t+1}\right)dt$$

$$=-\frac{1}{2}\int\frac{2t\,dt}{t^2+1}+\int\frac{dt}{t^2+1}+\frac{1}{2}\int\frac{(6t+2)dt}{3t^2+2t+1}-\int\frac{2dt}{\left[\sqrt{3}\left(t+\frac{1}{3}\right)\right]^2+\left(\sqrt{\frac{2}{3}}\right)^2}$$

$$=\frac{1}{2}\ln\left|t^2+1\right|+\tan^{-1}\left|t^2+1\right|+\frac{1}{2}\ln\left|3t^2+2t+1\right|-\sqrt{2}\tan^{-1}\left(\frac{3t+1}{\sqrt{2}}\right)+C$$

$$=\frac{1}{2}\ln\left|\frac{3-x}{2}\right|+\tan^{-1}\sqrt{\frac{1-x}{x-3}}+\frac{1}{2}\ln\left|2\sqrt{\frac{1-x}{x-3}}-\frac{2x}{x-3}\right|-\sqrt{2}\tan^{-1}\left(\frac{3\sqrt{1-x}-\sqrt{x-3}}{\sqrt{2(x-3)}}\right)+C$$

6. (d) Let $z=\frac{\sqrt{x^2+3x+2}}{x+1}$, $\sqrt{(x+1)(x+2)}=z(x+1)$, $x=-\frac{z^2-2}{z^2-1}, dx=-\frac{2z}{(z^2-1)^2}dz, \sqrt{(x+1)(x+2)}=z(x+1)=\frac{z}{z^2-1}$

$$\int\frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}dx=\int\frac{-2z^3-2z+4z}{(z+1)^3(z-1)^2(z-2)}dz=\int\left[\frac{5}{18(z+1)^2}+\frac{1}{3(z+1)^3}-\frac{17}{108(z+1)}+\frac{3}{4(z-1)}-\frac{16}{27(z-2)}\right]dz$$

$$=-\frac{5}{18(z+1)}-\frac{1}{6(z+1)^2}-\frac{17}{108}\ln|z+1|+\frac{3}{4}\ln|z-1|-\frac{16}{27}\ln|z-2|+C$$

$$=\frac{1}{108}\left\{-36x^2-60x+6(6x+1)\sqrt{x^2+3x+2}-64\ln|3x+2|-49\ln\left|2x+3+2\sqrt{x^2+3x+2}\right|+32\ln\left|5x+6+4\sqrt{x^2+3x+2}\right|\right\}+C$$

(e) Let $t^6=x$, $6t^5\,dt=dx$

$$\int\frac{dx}{x\left(1+2\sqrt{x}+\sqrt[3]{x}\right)}=\int\frac{6t^5\,dt}{t^6\left(1+2t^3+t^2\right)}=\int\frac{6dt}{t(t+1)\left(2t^2-t+1\right)}=\int\left[\frac{6}{t}-\frac{3/2}{t+1}-\frac{9t-3/2}{2t^2-t+1}\right]dt$$

$$=6\ln|t|-\frac{3}{2}\ln|t+1|-\frac{9}{4}\int\frac{4t-1}{2t^2-t+1}dt-\frac{3}{4}\int\frac{dt}{\left[\sqrt{2}(t-1/4)\right]^2+\left(\sqrt{7/8}\right)^2}$$

$$=6\ln|t|-\frac{3}{2}\ln|t+1|-\frac{9}{4}\ln\left|2t^2-t+1\right|-\frac{3}{2\sqrt{7}}\tan^{-1}\left(\frac{4t-1}{\sqrt{7}}\right)+C$$

$$=\frac{3}{4}\ln\left|\frac{x\sqrt[3]{x}}{\left(1+\sqrt[6]{x}\right)^2\left(1-\sqrt[6]{x}+2\sqrt[3]{x}\right)^3}\right|-\frac{3}{2\sqrt{7}}\tan^{-1}\left(\frac{4\sqrt[6]{x}-1}{\sqrt{7}}\right)+C$$

(f) and (g) Let $I_1=\int xe^x\cos x\,dx$, $I_2=\int xe^x\sin x\,dx$

$$I_1+iI_2=\int xe^x(\cos x+i\sin x)\,dx=\int xe^xe^{ix}\,dx=\int xe^{(1+i)x}\,dx=\frac{1}{1+i}\int x\,de^{(1+i)x}=$$

$$=\frac{1}{1+i}\left[xe^{(1+i)x}-\int e^{(1+i)x}dx\right]=\frac{1}{1+i}\left[xe^{(1+i)x}-\frac{e^{(1+i)x}}{1+i}\right]=\frac{e^{(1+i)x}}{(1+i)^2}[x(1+i)-1]$$

$$=\frac{e^x(\cos x+i\sin x)}{2i}[(x-1)+xi]=\frac{e^x}{2}(\cos x+i\sin x)[x-(x-1)i]$$

Compare real part and imaginary part,

$$I_1=\frac{e^x}{2}[x(\sin x+\cos x)-\sin x]+C_1,\qquad I_2=\frac{e^x}{2}[x(\sin x-\cos x)+\cos x]+C_2$$

(h) Let $u=x^2$, $dv=e^x\cos x\,dx$. Then $du=2x\,dx$, $v=\frac{1}{2}e^x(\sin x+\cos x)$, by 1(h).

$$I=\frac{x^2}{2}e^x(\sin x+\cos x)-\int xe^x(\sin x+\cos x)dx=\frac{x^2}{2}e^x(\sin x+\cos x)-\frac{e^x}{2}(2x\sin x+\cos x-\sin x)+C$$, by 6 (f), (g).

(i) First we let $C=\int xe^x\cos 2x\,dx$, $S=\int xe^x\sin 2x\,dx$

$$C+iS=\int xe^{x}(\cos 2x+i\sin 2x)dx=\int xe^{x}e^{2ix}dx=\int xe^{(1+2i)x}dx=\frac{1}{1+2i}\int x\ de^{(1+2i)x}=\frac{1}{1+2i}\left[xe^{(1+2i)x}-\int e^{(1+2i)x}dx\right]$$

$$=\frac{1}{1+2i}xe^{(1+2i)x}-\frac{1}{(1+2i)^2}e^{(1+2i)x}=\frac{1-2i}{5}xe^{(1+2i)x}+\frac{3+4i}{25}e^{(1+2i)x}=e^{x}(\cos 2x+i\sin 2x)\left[\frac{1-2i}{5}x+\frac{3+4i}{25}\right]$$

$$\therefore C=e^{x}\left[\frac{1}{5}x\cos 2x+\frac{2}{5}x\sin 2x+\frac{3}{25}\cos 2x-\frac{4}{25}\sin 2x\right]$$

$$\int xe^{x}\sin^2 x\ dx=\frac{1}{2}\int xe^{x}(1-\cos 2x)dx=\frac{1}{2}\int xe^{x}dx-\frac{1}{2}\int xe^{x}\cos 2xdx=\frac{1}{2}e^{x}[x-1]+C$$

$$=e^{x}\left[\frac{x-1}{2}-\frac{x}{10}(2\sin 2x+\cos 2x)+\frac{1}{50}(4\sin 2x-3\cos 2x))\right]+c$$

(j) First we let $C=\int x^2e^{x}\cos 2xdx,\quad S=\int x^2e^{x}\sin 2xdx$

$$C+iS=\int x^2e^{x}(\cos 2x+i\sin 2x)dx=\int x^2e^{x}e^{2ix}dx=\int x^2e^{(1+2i)x}dx=\frac{1}{1+2i}\int x^2\ de^{(1+2i)x}=\frac{1}{1+2i}\left[x^2e^{(1+2i)x}-2\int xe^{(1+2i)x}dx\right]$$

$$=\frac{1}{1+2i}x^2e^{(1+2i)x}-\frac{2}{(1+2i)^2}\int xde^{(1+2i)x}=\frac{1}{1+2i}x^2e^{(1+2i)x}-\frac{2}{(1+2i)^2}\left[xe^{(1+2i)x}-\int e^{(1+2i)x}dx\right]$$

$$=\frac{1}{1+2i}x^2e^{(1+2i)x}-\frac{2}{(1+2i)^2}xe^{(1+2i)x}+\frac{2}{(1+2i)^3}e^{(1+2i)x}=\frac{1-2i}{5}x^2e^{(1+2i)x}-\frac{6+8i}{25}xe^{(1+2i)x}-\frac{22-4i}{125}e^{(1+2i)x}$$

$$=e^{x}(\cos 2x+i\sin 2x)\left[\frac{1-2i}{5}x^2+\frac{6+8i}{25}x-\frac{22-4i}{125}\right]$$

Comparing real parts,

$$C=\frac{e^{x}}{125}\left[(25x^2+30x-22)\cos 2x+(50x^2-40x+4)\sin 2x\right]$$

$$\int x^2e^{x}\sin^2 x\ dx=\frac{1}{2}\int x^2e^{x}(1-\cos 2x)dx=\frac{1}{2}\int x^2e^{x}dx-\frac{1}{2}\int x^2e^{x}\cos 2xdx=\frac{1}{2}e^{x}[x^2-2x+2]-\frac{1}{2}C$$

$$=\frac{1}{2}e^{x}[x^2-2x+2]-\frac{e^{x}}{250}\left[(25x^2+30x-22)\cos 2x+(50x^2-40x+4)\sin 2x\right]+c$$

(k) Let $y=1+e^{x}$, $dy=e^{x}dx$, $dx=\dfrac{dy}{y-1}$

$$\int\frac{dx}{(1+e^{x})^2}=\int\frac{dy}{y^2(y-1)}=\int\left[\frac{1}{y-1}-\frac{1}{y}-\frac{1}{y^2}\right]dy=\ln|y-1|-\ln|y|+\frac{1}{y}+C=x-\ln|1+e^{x}|+\frac{1}{1+e^{x}}+C$$

(l) Let $y=e^{x/6}$, $dy=\dfrac{1}{6}e^{x/6}dx,\quad dx=\dfrac{6dy}{y}$

$$\int\frac{dx}{1+e^{x/2}+e^{x/3}+e^{x/6}}=\int\frac{6dy}{y(1+y^3+y^2+y)}=\int\frac{6dy}{y(1+y)(1+y^2)}=\int\left[\frac{6}{y}-\frac{3}{1+y}-\frac{3y+3}{1+y^2}\right]dy$$

$$=6\ln|y|-3\ln|1+y|-\frac{3}{2}\int\frac{2ydy}{1+y^2}-3\int\frac{dy}{1+y^2}=\ln y^6-3\ln|1+y|-\frac{3}{2}\ln|1+y^2|-3\tan^{-1}y+C$$

$$=x-3\ln|1+e^{x/6}|-\frac{3}{2}\ln\sqrt{1+e^{x/3}}-3\tan^{-1}e^{x/6}+C$$

(m) $\int\left(1-\frac{2}{x}\right)^2e^{x}dx=\int\left[1-\frac{4}{x}+\frac{4}{x^2}\right]e^{x}dx=\int\left[\left(1-\frac{4}{x}\right)de^{x}+e^{x}d\left(1-\frac{4}{x}\right)\right]=\int d\left[\left(1-\frac{4}{x}\right)e^{x}\right]=\left(1-\frac{4}{x}\right)e^{x}+C$

(n) $\int\left[\ln\left(x+\sqrt{1+x^2}\right)\right]^2dx=x\left[\ln\left(x+\sqrt{1+x^2}\right)\right]^2-\int 2\ln\left(x+\sqrt{1+x^2}\right)\frac{1}{x+\sqrt{1+x^2}}\left[1+\frac{x}{\sqrt{1+x^2}}\right]xdx$

$$= x\left[\ln\left(x+\sqrt{1+x^2}\right)\right]^2 - \int \frac{2x}{\sqrt{1+x^2}} \ln\left(x+\sqrt{1+x^2}\right)dx = x\left[\ln\left(x+\sqrt{1+x^2}\right)\right]^2 - \int \ln\left(x+\sqrt{1+x^2}\right)d\sqrt{1+x^2}$$

$$= x\left[\ln\left(x+\sqrt{1+x^2}\right)\right]^2 - 2\sqrt{1+x^2}\ln\left(x+\sqrt{1+x^2}\right) + 2\int \frac{1}{x+\sqrt{1+x^2}}\left[1+\frac{x}{\sqrt{1+x^2}}\right]\sqrt{1+x^2}\,dx$$

$$= x\left[\ln\left(x+\sqrt{1+x^2}\right)\right]^2 - 2\sqrt{1+x^2}\ln\left(x+\sqrt{1+x^2}\right) + 2x + C$$

(o) $$\int (x+|x|)^2\,dx = \int\left(x^2+2x|x|+|x|^2\right)dx = 2\int\left(x^2+x|x|\right)dx = 2\left[\frac{x^3}{3}+\frac{x^2|x|}{3}\right]+C = \frac{2}{3}\left[x^3+x^2|x|\right]+C$$

(p) $I = \int x\tan^{-1}x\ln\left(1+x^2\right)dx$

Let $u = \tan^{-1}x$, $du = \dfrac{dx}{1+x^2}$, $dv = x\ln\left(1+x^2\right)dx$, $v = \dfrac{1}{2}\int \ln\left(1+x^2\right)d\left(1+x^2\right) = \dfrac{1}{2}\left(1+x^2\right)\left[\ln\left(1+x^2\right)-1\right]$, by 1(c)

$$I = \frac{1}{2}\left(1+x^2\right)\left[\ln\left(1+x^2\right)-1\right]\tan^{-1}x - \frac{1}{2}\int\left(1+x^2\right)\left[\ln\left(1+x^2\right)-1\right]\frac{dx}{1+x^2}$$

$$= \frac{1}{2}\left(1+x^2\right)\left[\ln\left(1+x^2\right)-1\right]\tan^{-1}x - \frac{1}{2}\int \left[\ln\left(1+x^2\right)-1\right]dx$$

$$= \frac{1}{2}\left(1+x^2\right)\left[\ln\left(1+x^2\right)-1\right]\tan^{-1}x - \left\{\frac{x}{2}\left[\ln\left(1+x^2\right)-1\right] - \frac{1}{2}\int x\frac{2x}{1+x^2}dx\right\}$$

$$= \frac{1}{2}\left[\left(1+x^2\right)\tan^{-1}x - x\right]\left[\ln\left(1+x^2\right)-1\right] - \int\frac{\left(1+x^2\right)-1}{1+x^2}dx$$

$$= \frac{1}{2}\left[\left(1+x^2\right)\tan^{-1}x - x\right]\left[\ln\left(1+x^2\right)-1\right] - x + \tan^{-1}x + C$$

(r) Let $I = \int \dfrac{x^3\cos^{-1}x}{\sqrt{1-x^2}}\,dx$ $\quad u = \cos^{-1}x$, $\therefore du = -\dfrac{1}{\sqrt{1-x^2}}dx$, $\quad dv = \dfrac{x^3}{\sqrt{1-x^2}}dx$,

$$v = -\frac{1}{2}\int\frac{x^2}{\sqrt{1-x^2}}d\left(1-x^2\right) = \frac{1}{2}\int\frac{\left(1-x^2\right)-1}{\sqrt{1-x^2}}d\left(1-x^2\right) = \frac{1}{2}\int\left[\sqrt{1-x^2}-\frac{1}{\sqrt{1-x^2}}\right]d\left(1-x^2\right) = \frac{1}{2}\left[\frac{\left(1-x^2\right)^{3/2}}{3/2}-\frac{\left(1-x^2\right)^{1/2}}{1/2}\right]$$

$$= \frac{\sqrt{1-x^2}}{3}\left[\left(1-x^2\right)-3\right] = -\frac{\sqrt{1-x^2}}{3}\left(x^2+2\right)$$

$$\therefore I = -\frac{\sqrt{1-x^2}}{3}\left(x^2+2\right)\cos^{-1}x - \int\left[-\frac{\sqrt{1-x^2}}{3}\left(x^2+2\right)\right]\left[-\frac{dx}{\sqrt{1-x^2}}\right] = -\frac{\sqrt{1-x^2}}{3}\left(x^2+2\right)\cos^{-1}x - \frac{1}{3}\int\left(x^2+2\right)dx$$

$$= -\frac{\sqrt{1-x^2}}{3}\left(x^2+2\right)\cos^{-1}x - \frac{1}{3}\left[\frac{x^2}{3}+2x\right]+C, \qquad \text{where } |x|<1.$$

(s) $$\int\frac{\cos^3x+1}{\sin^2x}\,dx = \int\left[\frac{\cot^3x}{\csc x}+\csc^2x\right]dx = -\int\frac{\cot^2x}{\csc^2x}d(\csc x) + \int\csc^2x\,dx = \int\frac{1-\csc^2x}{\csc^2x}d(\csc x) - \cot x$$

$$= \int\left[\csc^{-2}x-1\right]d(\csc x) - \cot x = -\frac{1}{\csc x} - \csc x - \cot x + C = -\sin x - \csc x - \cot x + C$$

(t) $$\int\frac{a^x}{a^{2x}+1}\,dx = \int\frac{e^{x\ln a}}{e^{2x\ln a}+1}dx = \frac{1}{\ln a}\int\frac{de^{x\ln a}}{\left(e^{x\ln a}\right)^2+1} = \frac{1}{\ln a}\tan^{-1}\left(e^{x\ln a}\right) = \frac{\tan^{-1}a^x}{\ln a}+C$$

(u) $$\int e^{\sqrt{x}}dx = 2\int\sqrt{x}\,de^{\sqrt{x}} = 2\left[\sqrt{x}e^{\sqrt{x}} - \int\frac{e^{\sqrt{x}}}{2\sqrt{x}}dx\right] = 2\left[\sqrt{x}e^{\sqrt{x}} - e^{\sqrt{x}}\right]+C$$

(v) $I = \int\dfrac{dx}{1+\sqrt{x}+\sqrt{1+x}}$, $\qquad$ Put $x = \left(\dfrac{u^2-1}{2u}\right)^2$ $\qquad dx = \dfrac{\left(u^2-1\right)^2}{2u^3}du$

Also, $u^2 - 2u\sqrt{x} - 1 = 0$, $\qquad u = \sqrt{x}+\sqrt{1+x}$

$$I=\int\frac{1}{1+u}\frac{(u^2-1)^2}{2u^3}du=\int\frac{(u+1)(u-1)^2}{2u^3}du=\int\frac{u^3-u^2-u+1}{2u^3}du=\frac{1}{2}\left[u-\ln u+\frac{1}{u}-\frac{1}{u^2}\right]+C$$

$$=\frac{1}{2}\left[\left(\sqrt{x}+\sqrt{1+x}\right)-\ln\left(\sqrt{x}+\sqrt{1+x}\right)+\frac{1}{\sqrt{x}+\sqrt{1+x}}-\frac{1}{\left(\sqrt{x}+\sqrt{1+x}\right)^2}\right]+C$$

(w) $\int x|x|\ dx=\frac{1}{3}x^2|x|+C$

(x) $I=\int\max(1,x^2)\ dx=\begin{cases}\int dx=x+C & ,\text{when } |x|\le 1\\ \int x^2dx=\frac{x^3}{3}+C & ,\text{when } |x|>1\end{cases}$

(y) $\int\{|1+x|-|1-x|\}\,dx=\frac{(1+x)|1+x|}{2}+\frac{(1-x)|1-x|}{2}+C$

(z)

$$\int_{1/e}^{e}|\log_{10}x|\,dx=\frac{1}{\ln 10}\int_{1/e}^{e}|\ln x|\,dx=\frac{1}{\ln 10}\left[\int_1^e\ln x\,dx-\int_{1/e}^{1}\ln x\,dx\right]=\frac{1}{\ln 10}\left[(x\ln x-x)\Big|_1^e-(x\ln x-x)\Big|_{1/e}^{1}\right]=\frac{2}{\ln 10}\left(1-\frac{1}{e}\right)$$

7. (a) $\int\frac{dx}{x^2(1+x)}=\int\left[\frac{1}{x^2}-\frac{1}{x}+\frac{1}{1+x}\right]dx=-\frac{1}{x}-\ln|x|+\ln|x+1|+C=\ln\left|1+\frac{1}{x}\right|-\frac{1}{x}+C$

(b) For $I=\int_0^a f(a-x)\,dx$, let $y=a-x$. When $x=a, y=0$; when $x=0$, $y=a$.

$\therefore I=\int_a^0 f(y)(-dy)=\int_0^a f(y)\,dy=\int_0^a f(x)\,dx$.

$$I=\int_0^{\pi/2}\frac{x\,dx}{\sin x+\cos x}=\int_0^{\pi/2}\frac{\left(\frac{\pi}{2}-x\right)dx}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}=\int_0^{\pi/2}\frac{\left(\frac{\pi}{2}-x\right)dx}{\sin x+\cos x}=\frac{\pi}{2}-\int_0^{\pi/2}\frac{dx}{\sin x+\cos x}=\frac{\pi}{2}-I$$

$$\therefore I=\int_0^{\pi/2}\frac{dx}{\sin x+\cos x}=\int_0^{\pi/2}\frac{dx}{\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)}=\frac{1}{\sqrt{2}}\int_0^{\pi/2}\sec\left(x-\frac{\pi}{4}\right)dx=\frac{1}{\sqrt{2}}\ln\left|\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right|_0^{\pi/2}=\frac{\pi}{2\sqrt{2}}\ln\left(\sqrt{2}+1\right)$$

8. $I=\int_a^b\frac{dx}{\sqrt{(x-a)(b-x)}}=\pi$, by 4(k)

Let $a=e^{-\beta}$, $b=e^{-\alpha}$, since $\alpha<\beta$, $b>a>0$.

Put $x=e^{-\theta}$, $dx=e^{-\theta}d\theta$. When $x=a,\theta=\beta$; $x=b,\theta=\alpha$.

$$\pi=\int_a^b\frac{dx}{\sqrt{(x-a)(b-x)}}=\int_\beta^\alpha\frac{-e^{-\theta}d\theta}{\sqrt{(e^{-\theta}-e^{-\beta})(e^{-\alpha}-e^{-\theta})}}=e^{\frac{1}{2}(\alpha+\beta)}\int_\alpha^\beta\frac{d\theta}{\sqrt{\left(e^\theta-e^\alpha\right)\left(e^\beta-e^\theta\right)}}$$

$$\int_\alpha^\beta\frac{d\theta}{\sqrt{\left(e^\theta-e^\alpha\right)\left(e^\beta-e^\theta\right)}}=\pi e^{-\frac{1}{2}(\alpha+\beta)}$$

9. (i) $\int_{1/x}^{x}\frac{x^2}{1+x^3}dx=\left[\frac{1}{3}\ln\left|1+x^3\right|\right]_{1/x}^{x}=\frac{1}{3}\left[\ln\left|1+x^3\right|-\ln\left|1+\frac{1}{x^3}\right|\right]=\ln|x|$

(ii) For $I=\int_{1/\alpha}^{\alpha}\frac{1}{1+x^3}dx$. Let $x=\frac{1}{y}, dx=-\frac{1}{y^2}dy$,

$$I=\int_\alpha^{1/\alpha}\frac{1}{1+(1/y)^3}\left(-\frac{1}{y^2}dy\right)=\int_{1/\alpha}^{\alpha}\frac{y}{1+y^3}dxy=\int_{1/\alpha}^{\alpha}\frac{x}{1+x^3}dx$$

(iii) $\int_{1/2}^{2}\frac{1}{1+x^3}dx=\frac{1}{2}\left[\int_{1/2}^{2}\frac{1}{1+x^3}dx+\int_{1/2}^{2}\frac{x}{1+x^3}dx\right]=\frac{1}{2}\int_{1/2}^{2}\frac{1+x}{1+x^3}dx=\frac{1}{2}\int_{1/2}^{2}\frac{1}{1-x+x^2}dx$

$=\frac{1}{2}\int_{1/2}^{2}\frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\sqrt{\frac{3}{4}}\right)^2}dx=\left[\sqrt{\frac{3}{4}}\tan^{-1}\left(x-\frac{1}{2}\right)\right]_{1/2}^{2}=\frac{\pi}{3\sqrt{3}}$

(iv) $\int_{1/\alpha}^{\beta}\frac{1}{1+x^3}dx \neq \int_{1/\alpha}^{\beta}\frac{x}{1+x^3}dx,\qquad \int_{1/\alpha}^{\beta}\frac{1}{1+x^3}dx=\int_{1/\beta}^{\alpha}\frac{x}{1+x^3}dx$

10. $\int\frac{dx}{\sin(x+a)\sin(x+b)}=\frac{1}{\sin(a+b)}\int\frac{\sin(a+b)dx}{\sin(x+a)\sin(x+b)}=\frac{1}{\sin(a+b)}\int\frac{\sin(x+a)\cos(x+b)-\cos(x+a)\sin(x+b)dx}{\sin(x+a)\sin(x+b)}$

$=\frac{1}{\sin(a+b)}\int\left[\frac{\cos(x+b)}{\sin(x+b)}-\frac{\cos(x+a)}{\sin(x+a)}\right]dx=\frac{1}{\sin(a+b)}\left[\int\frac{d\sin(x+b)}{\sin(x+b)}-\int\frac{d\sin(x+a)}{\sin(x+a)}\right]=\frac{1}{\sin(a+b)}\ln\left|\frac{\sin(x+b)}{\sin(x+a)}\right|+C$

Similarly,

$\int\frac{dx}{\sin(x+a)\cos(x+b)}=\frac{1}{\cos(a-b)}\ln\left|\frac{\sin(x+a)}{\cos(x+b)}\right|+C,\qquad \int\frac{dx}{\cos(x+a)\cos(x+b)}=\frac{1}{\sin(a-b)}\ln\left|\frac{\cos(x+b)}{\cos(x+a)}\right|+C$

11. Let $I=(b-a)\int_0^1 f[a+(b-a)x]\,dx$,

Let $y=a+(b-a)x$. $dy=(b-a)dx$, $dx=\frac{1}{b-a}dy$

$I=(b-a)\int_a^b f(y)\frac{dy}{b-a}=\int_a^b f(y)\frac{dy}{b-a}=\int_a^b f(x)dx$

12. (a) $I=\int_1^2\frac{x\,dx}{x^2+1}=\frac{\varepsilon}{\varepsilon^2+1}\int_1^2 dx=\frac{\varepsilon}{\varepsilon^2+1}$, by Mean Value Theorem, where $1\le\varepsilon\le 2$.

When $\varepsilon=2,\frac{\varepsilon}{\varepsilon^2+1}=\frac{2}{5}$, When $\varepsilon=1,\frac{\varepsilon}{\varepsilon^2+1}=\frac{1}{2}$

Since $\frac{\varepsilon}{\varepsilon^2+1}$ is a decreasing function for $1\le\varepsilon\le 2$, therefore $\frac{2}{5}<\int_1^2\frac{x\,dx}{x^2+1}<\frac{1}{2}$.

(b) $I=\int_0^1 e^{x^2}dx=e^{\varepsilon^2}\int_0^1 dx=e^{\varepsilon^2}$, by Mean Value Theorem, where $0\le\varepsilon\le 1$.

When $\varepsilon=1, e^{\varepsilon^2}=e$, When $\varepsilon=0, e^{\varepsilon^2}=1$

Since e^{ε^2} is a increasing function for $0\le\varepsilon\le 1$, therefore $1<\int_0^1 e^{x^2}dx<e$.

(c) $I=\int_0^{\pi/2}\frac{\sin x}{x}dx=\frac{\sin\varepsilon}{\varepsilon}\int_0^{\pi/2}dx=\frac{\sin\varepsilon}{\varepsilon}\times\frac{\pi}{2}$, by Mean Value Theorem, where $0\le\varepsilon\le\frac{\pi}{2}$.

When $\varepsilon=\frac{\pi}{2},\frac{\sin\varepsilon}{\varepsilon}\times\frac{\pi}{2}=1$; When $\varepsilon\to 0,\frac{\sin\varepsilon}{\varepsilon}\times\frac{\pi}{2}\to\frac{\pi}{2}$

Since $\frac{\sin\varepsilon}{\varepsilon}$ is a increasing function for $0\le\varepsilon\le\frac{\pi}{2}$, therefore $1<\int_0^{\pi/2}\frac{\sin x}{x}dx<\frac{\pi}{2}$.